数理解析研究所講究録
第 1651 巻 2009 年 132-136

Some results on quotient Aubry sets

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December 19, 2008

1 Introduction

This note consists of partial results of my recent paper [Fu]. In [Fu], we discuss several topics which are not treated here.

Let Ω be an open and connected subset of $\mathbb{R}^n$, and $H : \Omega\times\mathbb{R}^n \to \mathbb{R}$ a given function. In this paper, we consider the Hamilton-Jacobi equation

$$H(x, Du(x)) = 0 \quad \text{in } \Omega. \tag{1.1}$$

Let $\mathcal{A}$ and $\overline{\mathcal{A}}$ be, respectively, its (projected) Aubry set and the quotient Aubry set. As for their definitions and properties, see Section 2 below. The quotient Aubry set $\overline{\mathcal{A}}$ plays an essential role to study viscosity solutions of (1.1) (cf. [CI, DS, FU, I, IM]). In particular, several authors provided sufficient conditions in order that $\overline{\mathcal{A}}$ is totally disconnected (i.e., every connected component consists of a single point in the topology of $\overline{\mathcal{A}}$) [FFR, M1, M2, S].

In this note, we explain a reason why total disconnectedness of $\overline{\mathcal{A}}$ is important. Let $\pi(x)$ be the equivalent class of $\overline{\mathcal{A}}$ containing $x\in\mathcal{A}$. We study how $\pi(x)$ behaves in $\mathcal{A}$ when $\overline{\mathcal{A}}$ is total disconnected. We show that a necessary condition in order that $\overline{\mathcal{A}}$ is totally disconnected is that $\pi(x)\supset C(x)$ holds for each $x\in\mathcal{A}$. Here, $C(x)$ is the connected component of $\mathcal{A}$ containing $x\in\mathcal{A}$. On the other hand, we show that if $\mathcal{A}$ is a compact set in Ω, then a necessary and sufficient condition in order that $\overline{\mathcal{A}}$ is totally disconnected is that $\pi(x) = C(x)$ holds for each $x\in\mathcal{A}$.

The state such that $\pi(x) = C(x)$ for each $x\in\mathcal{A}$ is preferable, because we can understand and calculate $\pi(x)$ of this case clearly. Our result shows that if $\mathcal{A}$ is a compact set in Ω, this preferable state occurs when and only when $\overline{\mathcal{A}}$ is totally disconnected. This is a reason why we propose that totally disconnectedness of $\overline{\mathcal{A}}$ is important.

The contents of this note are as follows: In Section 2, we provide some preliminaries. In Section 3, we state our results .

*Supported in part by Grant-in-Aid for Scientific Research, No. 18540165, 18204009, JSPS.

2 Preliminaries

Let $B(x,r)=\{y\in\mathbb{R}^n\,|\,|y-x|\le r\}$ for $x\in\mathbb{R}^n$ and $r>0$. We assume:

(A1) $H\in C(\Omega\times\mathbb{R}^n)$.

(A2) H is coercive, that is, for any compact subset K of Ω,

$$\lim_{r\to\infty}\inf\{H(x,p)\,|\,x\in K,\ p\in\mathbb{R}^n\setminus B(0,r)\}=\infty.$$

(A3) For any $x\in\Omega$, the function $p\mapsto H(x,p)$ is convex on $\mathbb{R}^n$.

(A4) There is a continuous viscosity subsolution of (1.1).

Let $\mathcal{S}$ (resp., $\mathcal{S}^-$) denotes the space of continuous viscosity solutions (resp., viscosity subsolutions) of (1.1). If necessary, we write $\mathcal{S}(\Omega)$ and $\mathcal{S}^-(\Omega)$ for $\mathcal{S}$ and $\mathcal{S}^-$, respectively, in order to refer the domain under consideration. Then, (A4) implies that $\mathcal{S}^-(\Omega)\ne\emptyset$.

Next, we explain the (projected) Aubry set for the Hamilton-Jacobi equation (1.1). The Aubry set is defined as follows: Define the function $d:\Omega\times\Omega\mapsto(-\infty,\infty]$ by

$$d(x,y)=\sup\left\{v(x)-v(y)\,|\,v\in\mathcal{S}^-(\Omega)\right\}. \tag{2.1}$$

Then, by [IM, Theorem 1.4 and Proposition 1.6], we have the following:

(2.2) d is locally Lipschitz continuous on $\Omega\times\Omega$.

(2.3) $u(x)-u(y)\le d(x,y)$ for all $u\in\mathcal{S}^-(\Omega)$ and $x,y\in\Omega$.

(2.4) For all $y\in\Omega$, $d(\cdot,y)\in\mathcal{S}^-(\Omega)$ and $d(\cdot,y)\in\mathcal{S}(\Omega\setminus\{y\})$.

(2.5) For all $x,y,z\in\Omega$, $d(x,z)\le d(x,y)+d(y,z)$ and $d(x,x)=0$.

$$d(x,y)=\inf\left\{\int_0^t L(\gamma(s),\dot\gamma(s))ds\,\middle|\,t>0,\ \gamma\in\mathcal{C}(x,t;y,0)\right\}, \tag{2.6}$$

where $\mathcal{C}(x,t;y,0)$ is the set of all absolutely continuous curves $\gamma:[0,t]\mapsto\Omega$ satisfying $(\gamma(t),\gamma(0))=(x,y)$, and $L\in C(\Omega\times\mathbb{R}^n)$ is the convex conjugate of H defined by

$$L(x,\xi)=\sup\{\xi\cdot p-H(x,p)\,|\,p\in\mathbb{R}^n\}\quad\text{for }(x,\xi)\in\Omega\times\mathbb{R}^n. \tag{2.7}$$

The Aubry set $\mathcal{A}$ is defined by

$$\mathcal{A}=\{y\in\Omega\,|\,d(\cdot,y)\in\mathcal{S}(\Omega)\}. \tag{2.8}$$

In the following, we assume

(A5) $\mathcal{A}\ne\emptyset$.

We note that $\mathcal{A}$ is a closed set in Ω, which is due to the stability of the viscosity property under uniform convergence. The assumptions (A1)-(A5) are considered to be natural to discuss the Aubry set for the Hamilton-Jacobi equation (1.1).

Now, we explain an equivalence relation on $\mathcal{A}$, which is important to study $\mathcal{S}(\Omega)$ and $\mathcal{S}^-(\Omega)$. By (2.5), we see that the function $\lambda : \mathcal{A} \times \mathcal{A} \to \mathbb{R}$ defined by $\lambda(x,y) = d(x,y) + d(y,x)$ is a pseudo-metric on $\mathcal{A}$, i.e., it is non-negative, symmetric, and satisfies the triangle inequality and $\lambda(x,x) = 0$; but the condition $\lambda(x,y) = 0$ does not necessarily imply $x = y$. Let $x, y \in \mathcal{A}$. If $\lambda(x,y) = 0$, then we write $x\delta y$. This relation δ is an equivalence relation on $\mathcal{A}$. We set

$$\begin{aligned} \pi(y) &= \{z \in \mathcal{A} \mid z\delta y\}, \quad y \in \mathcal{A}, \\ \overline{\mathcal{A}} &= \{\pi(y) \mid y \in \mathcal{A}\}. \end{aligned} \tag{2.9}$$

Then, π is considered as the canonical surjection from $\mathcal{A}$ to $\overline{\mathcal{A}}$, and we see that $\overline{\mathcal{A}} = \mathcal{A}/\delta$. Note that we may regard $\xi \in \overline{\mathcal{A}}$ as a subset of $\mathcal{A}$ and $\xi = \pi^{-1}(\{\xi\})$. Note also that if $x \in \pi(y)$, then $\pi(x) = \pi(y)$. We define the function $\overline{\lambda} : \overline{\mathcal{A}} \times \overline{\mathcal{A}} \to \mathbb{R}$ by

$$\overline{\lambda}(\pi(x), \pi(y)) = d(x,y) + d(y,x). \tag{2.10}$$

The following proposition is well-known.

Proposition 1. *$\overline{\lambda}$ is well defined, and $(\overline{\mathcal{A}}, \overline{\lambda})$ is a metric space.*

3 Results

In this section, we state our results of this note. For their proofs, see [Fu]. Let $C(x)$ be the connected component of $\mathcal{A}$ containing x. In the following, as the topology of $\overline{\mathcal{A}}$, we always consider the one induced by the metric $\overline{\lambda}$. Note that, by (2.2) and (2.10), π is a continuous mapping from $\mathcal{A}$ to $\overline{\mathcal{A}}$.

Proposition 2. *Assume (A1)-(A5). If $\overline{\mathcal{A}}$ is totally disconnected, then $\pi(x) \supset C(x)$ for each $x \in \mathcal{A}$.*

By Proposition 2, the condition that $\pi(x) \supset C(x)$ for each $x \in \mathcal{A}$ is a necessary condition in order that $\overline{\mathcal{A}}$ is totally disconnected. Next, we consider a sufficient condition in order that $\overline{\mathcal{A}}$ is totally disconnected. In the following, we assume:

(A6) $\mathcal{A}$ is a compact set of Ω.

We provide a simple consequence of (A6).

Lemma 1. *Assume (A1)-(A6). Then, $\pi(x)$ is a connected set of $\mathcal{A}$ for each $x \in \mathcal{A}$.*

Now, we are in the position to state our sufficient condition in order that $\overline{\mathcal{A}}$ is totally disconnected.

Proposition 3. *Assume that (A1)-(A6). Then, $\overline{\mathcal{A}}$ is totally disconnected if and only if $\pi(x) = C(x)$ for each $x \in \mathcal{A}$.*

Acknowledgments. I would like to thank Professor Antonio Siconolfi for giving a nice lecture on their paper [FS2] at the Fukuoka University in October 2006, and Professor Naoki Yamada for inviting me to this lecture. This lecture is very helpful to write the manuscript.

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